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Multivariate Divided Differences and Multivariate Interpolation of Lagrange and Hermite Type

HAKOP A. HAKOPIAN

Institute of Mathematics, Gdańsk University, Gdańsk, Poland and
Department of Mathematics, University of Yerevan, Yerevan, Armenian SSR, USSR

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We give a natural definition of multivariate divided differences and we construct the multivariate analog of Lagrange interpolation. We consider Hermite interpolation in the plane case only. We also give a multivariate representation of a function f in terms of the above mentioned interpolating polynomials and divided differences.

INTRODUCTION

In this paper we present a natural definition of multivariate divided differences. We also generalize Lagrange and Hermite interpolation to the multivariate case. This gives a linear projection from $C^0(R^k)$ (the space of continuous functions on R^k) onto $\Pi_{r-k+1}(R^k)$ (the space of k-variate polynomials of total degree $\leqslant r-k+1$), where $r+1$ is the number of interpolation "knots".

For another, closely related, approach to multivariate Lagrange–Hermite interpolation, namely, Kergin interpolation, we refer to [2, 9–12]. Kergin interpolation in the Lagrange case gives a linear projection from the space $C^{k-1}(R^k)$ onto $\Pi_r(R^k)$.

The authors in [2] gave a related but different definition of multivariate divided differences, suitable for Kergin's approach.

Some basic formulas presented here were anounced in [6]. They are analogous to the one-dimensional ones (see [3]).

In our investigation multivariate B-splines play important role. They were introduced by de Boor (see [1]) who followed the geometric interpretation of the univariate B-splines given by H. B. Curry and I. J. Schoenberg (see [4]). The recurrence relations of Micchelli for the multivariate B-splines and the related linear functionals we shall use often.

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1. MULTIVARIATE B-SPLINES

We begin with the definition of the univariate divided differences for distinct points:

$$[t_0,..., t_r]\, f(t) := \sum_{i=0}^{r} \frac{f(t_i)}{(t_i - t_0) \cdots (t_i - t_{i-1})(t_i - t_{i+1}) \cdots (t_i - t_r)}.$$

It is easy to check the following useful relation

$$[t_0,..., t_r](t - t_j)\, f(t) = [t_0,..., t_{j-1}, t_{j+1},..., t_r]\, f(t), \qquad 0 \leqslant j \leqslant r. \tag{1}$$

From this relation we readily get the familiar recurrence relation for the divided difference

$$[t_0,..., t_r]\, f(t) = \frac{[t_0,..., t_{r-1}]\, f(t) - [t_1,..., t_r]\, f(t)}{t_0 - t_r}. \tag{2}$$

The Hermite–Genocchi representation for the divided difference of a smooth function is

$$[t_0,..., t_r]\, f(t) = \int_{Q^r} f^{(r)}(v_0 t_0 + \cdots + v_r t_r)\, dv_1 \cdots dv_r, \tag{3}$$

where $Q^r = \{(v_1,..., v_r) \mid \sum_{i=1}^r v_i \leqslant 1, v_j \geqslant 0, j = 1,..., r\}$ and $v_0 = 1 - \sum_{i=1}^r v_i$.

To prove (3) (see [13]) it is enough to check that the right hand side integral in (3), as well as the left hand side, satisfies the recurrence relation (2).

Let us denote as in [12]

$$\int_{[x^0,\ldots,x^r]} f := \int_{Q^r} f(v_0 x^0 + \cdots + v_r x^r)\, dv_1 \cdots dv_r, \tag{4}$$

where $x^i \in R^k$, $i = 0,..., r$ and $f: R^k \to R$.

Now on account of relations (1), (3) we have

$$\int_{[t_0,\ldots,t_r]} ((t - t_j)\, f(t))^{(r)} = \int_{[t_0,\ldots,t_{j-1},t_{j+1},\ldots,t_r]} f^{(r-1)}.$$

If we set $f^{(r-1)} := \varphi$, we get

$$\int_{[t_0,\ldots,t_r]} (t - t_j)\, \varphi'(t) + r \cdot \int_{[t_0,\ldots,t_r]} \varphi = \int_{[t_0,\ldots,t_{j-1},t_{j+1},\ldots,t_r]} \varphi.$$

This relation is easily generalized to multivariate functions $f\colon R^k \to R$ and $x^i \in R^k$, $i = 0,\dots, r$.

Explicitly, we have

$$\int_{[x^0,\dots,x^r]} D_{x-x^j} f(x) + r\int_{[x^0,\dots,x^r]} f = \int_{[x^0,\dots,x^{j-1},x^{j+1},\dots,x^r]} f. \tag{5}$$

To prove this it is enough to check it for so called "ridge" functions

$$f(x) = f(x_1,\dots, x_k) = g(\lambda_1 x_1 + \cdots + \lambda_k x_k) = g(\lambda x),$$

where g is a univariate function (for this method see [9]).

For "ridge" function f we have

$$\int_{[x^0,\dots,x^r]} f = \int_{[\lambda x^0,\dots,\lambda x^r]} g \qquad \text{and} \qquad D_{x-x^j} f(x) = (\lambda x - \lambda x^j)\, g'(\lambda x).$$

Therefore (5) is reduced to the univariate case.

Now let $y = \sum_{i=0}^r \mu_i x^i$, $\mu = \sum_{i=0}^r \mu_i$. Then we find from (5) the relation

$$\int_{[x^0,\dots,x^r]} D_{\mu x - y} f(x) + r\mu \int_{[x^0,\dots,x^r]} f = \sum_{i=0}^r \mu_i \int_{[x^0,\dots,x^{i-1},x^{i+1},\dots,x^r]} f. \tag{6}$$

In particular if $\mu = 0$, then (6) reduced to the following Micchelli's relation (cf. [12])

$$\int_{[x^0,\dots,x^r]} D_y f = -\sum_{i=0}^r \mu_i \int_{[x^0,\dots,x^{i-1},x^{i+1},\dots,x^r]} f. \tag{7}$$

We now recall the definition of the k-variate B-spline with knot set $\{x^0,\dots, x^r\}$, $r \geqslant k+1$, $\mathrm{vol}_k [x^0,\dots, x^r] \neq 0$, where

$$[x^0,\dots, x^r] := \left\{ x \;\middle|\; x = \sum_{i=0}^r v_i x^i,\ \sum_{i=0}^r v_i = 1,\ v_j \geqslant 0,\ j = 0,\dots, r \right\}.$$

The condition imposed on $x^0,\dots, x^r$ implies existence of a proper simplex $\sigma = [y^0,\dots, y^r]$ in R^k with vertices y^i, $i = 0,\dots, r$ having the same first k coordinates as x^i, $i = 0,\dots, r$, respectively (see [5]), that is, $y^i \in R^r$, $i = 0,\dots, r$, $y^0 = (x^0,\dots),\dots, y^r = (x^r,\dots)$ and $\mathrm{vol}_r\, \sigma \neq 0$, $\sigma = [y^0,\dots, y^r]$.

Now de Boor's definition of the k-variate B-spline at $x \in R^k$, $x = (x_1,\dots, x_k)$ is

$$M(x \mid x^0,\dots, x^r) = \frac{\mathrm{vol}_{r-k}\{ y \in \sigma \mid y_j = x_j,\ j = 1,\dots, k\}}{\mathrm{vol}_r\, \sigma}.$$

In one dimension this is just the Curry–Schoenberg geometric interpretation of univariate B-spline [4].

The following important relation implies the independence of $M(x \mid x^0,..., x^r)$ from the choice of y^i, $i = 0,..., r$.

$$\int_{R^k} f(x)\, M(x \mid x^0,..., x^r)\, dx = r! \int_{[x^0,...,x^r]} f. \tag{8}$$

The proof of this relation is based on familiar change of variables $\mathbf{t} = (t_1,..., t_r) = v_0 y^0 + \cdots + v_r y^r$.

Still to have (8) also for $r = k$, let us define

$$M(x \mid x^0,..., x^k) = \frac{\chi_\sigma}{\operatorname{vol}_k \sigma},$$

where $\sigma = [x^0,..., x^k]$ and χ_σ is the characteristic function for σ.

Combining relations (5) and (8) we get

$$\int_{R^k} M(x \mid x^0,..., x^r)\, D_{x-x^j} f(x)\, dx + r \int_{R^k} M(x \mid x^0,..., x^r)\, f(x)\, dx$$
$$= \int_{R^k} M(x \mid x^0,..., x^{j-1}, x^{j+1},..., x^r)\, f(x)\, dx.$$

If the splines $M(x \mid x^0,..., x^r)$ and $M(x \mid x^0,..., x^{j-1}, x^{j+1},..., x^r)$ are continuous then by a standard argument $M(x \mid x^0,..., x^r)$ has continuous partial derivatives and

$$D_{x^j - x} M(x \mid x^0,..., x^r) + (r-k)\, M(x \mid x^0,..., x^r)$$
$$= r \cdot M(x \mid x^0,..., x^{j-1}, x^{j+1},..., x^r). \tag{9}$$

This relation was presented in [7] in greater generality, here we have proved it by the method similar to Micchelli's in [13].

From (9) we easily get for $y = \sum_{i=0}^r \mu_i x^i$ and $\mu = \sum_{i=0}^r \mu_i$.

$$D_{y-\mu x} M(x \mid x^0,..., x^r) + \mu(r-k)\, M(x \mid x^0,..., x^r)$$
$$= r \sum_{i=0}^{r} \mu_i M(x \mid x^0,..., x^{i-1}, x^{i+1},..., x^r). \tag{10}$$

We mention that two important special cases of (10), namely, the cases (i) $\mu = 0$, (ii) $\mu = 1$, $x = y = \sum_{i=0}^r \mu_i x^i$, were found earlier by Micchelli [12] and (i) was independently found by Dahmen [5].

We have proved here relations (9), (10) under the condition that all the B-splines appearing there are continuous. Of course the latter is the case if the points $x^0,\dots,x^r$, $r \geqslant k+1$ are in general position, that is every $k+1$ points from $\{x^0,\dots,x^r\}$ are affinely independent. In spite of this conjugate relations (5), (6) and (7) hold without restrictions.

Finally we present Micchelli's recurrence relation (see [13]) which in the univariate case is due to Meinardus:

$$M(x \mid x^0,\dots,x^r) = r \int_1^\infty t^{-r+k-1} M((1-t)\,x^0 + tx \mid x^1,\dots,x^r)\,dt, \qquad (11)$$

whenever $\mathrm{vol}_k[x^1,\dots,x^r] \neq 0$.

2. Multivariate Divided Differences

The following definition differs from the one presented in [2] on the value of modulus of the multi-integer $\alpha = (\alpha_1,\dots,\alpha_k)$.

Definition. Let $x^0,\dots,x^r \in R^k$, $\mathrm{vol}_k[x^0,\dots,x^r] \neq 0$, $\alpha = (\alpha_1,\dots,\alpha_k)$, $|\alpha| = \alpha_1 + \cdots + \alpha_k = r-k+1$ and let f be sufficiently smooth. Then the k-variate α-divided difference of the function f at $x^0,\dots,x^r$ is

$$[x^0,\dots,x^r]^\alpha f := \frac{1}{\alpha!}\int_{R^k} M(x \mid x^0,\dots,x^r)\, D^\alpha f(x)\,dx,$$

where

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_k}\right)^{\alpha_k}, \qquad \alpha! = \alpha_1! \cdots \alpha_k!,$$

and

$$[x^0,\dots,x^{k-1}]^\alpha f := (k-1)! \int_{[x^0,\dots,x^{k-1}]} f =: f\{x^0,\dots,x^{k-1}\}.$$

Let us now introduce some notation: $I_m^n :=$ collection of subsets of $\{0,\dots,n\}$ of cardinality m.

We briefly write, for $i = (i_0,\dots,i_{k-1}) \in I_k^r$,

$$\{x^i\} := \{x^{i_0},\dots,x^{i_{k-1}}\}.$$

We set

$$d(x^i, y, \gamma) := \begin{vmatrix} y_1 & x_1^{i_0} & \cdots & x_1^{i_{k-1}} \\ \vdots & \vdots & & \vdots \\ y_k & x_k^{i_0} & \cdots & x_k^{i_{k-1}} \\ \gamma & 1 & & 1 \end{vmatrix}$$

for $y = (y_1,\dots,y_k) \in R^k$, $i = (i_0,\dots,i_{k-1}) \in I_k^r$, $\gamma \in R$.

Let also $e^l \in R^k$, $(e^l)_j = \delta_j^l$, $j = 1,\dots,k$.

THEOREM 1. *Let* $x^0,\ldots,x^r \in R^k$ *be in general position, that is,* $\mathrm{vol}_k[x^{j_0},\ldots,x^{j_k}] \neq 0$ *for all* $(j_0,\ldots,j_k) \in I^r_{k+1}$, $\alpha = (\alpha_1,\ldots,\alpha_k)$, $|\alpha| = r-k+1$.
Then

$$[x^0,\ldots,x^r]^\alpha f = \sum_{i \in I^r_k} C_i^\alpha f\{x^i\},$$

where

$$C_i^\alpha = (-1)^{r-k+1} \frac{r!}{\alpha!\,(k-1)!} \frac{\prod_{l=1}^k d(x^i, e^l, 0)}{\prod_{l \neq i_0,\ldots,i_{k-1}} d(x^i, x^l, 1)}$$

for $i = (i_0,\ldots,i_{k-1})$. *Also* $f\{x^i\} := f\{x^{i_0},\ldots,x^{i_{k-1}}\}$.

The following Lemma 1 is interesting in itself and it is crucial in the proof of Theorem 1. To establish it we need some preliminaries.

Let $\mathrm{vol}_k[x^0,\ldots,x^r] \neq 0$. Regions which are bounded but not intersected by the convex hull of k points from $\{x^0,\ldots,x^r\}$ we shall call p-regions.

It is not difficult to prove (with the help of (10)) that $M(x \mid x^0,\ldots,x^r)$ is a polynomial of total degree $\leqslant r-k$ in every p-region and, if $x^0,\ldots,x^r$ are in general position then $M(x \mid x^0,\ldots,x^r) \in C^{r-k+1}(R^k)$ (see [5, 7, 12]).

Thus $D^\beta M(x \mid x^0,\ldots,x^r)$, $|\beta| = r-k$, is constant on every p-region E, let us denote it by $D^\beta M/E$.

If E_1, E_2 are neighbouring p-regions with common side contained in $[x^{i_0},\ldots,x^{i_{k-1}}]$ then we set

$$\Delta_i \big|_{E_2}^{E_1} D^\beta M := \Delta_{[x^{i_0},\ldots,x^{i_{k-1}}]} \big|_{E_2}^{E_1} D^\beta M(x \mid x^0,\ldots,x^r) := D^\beta M\big|_{E_1} - D^\beta M\big|_{E_2}.$$

By the right hand side (left hand side) of $[x^{i_0},\ldots,x^{i_{k-1}}]$ we mean the half-space $\{x \mid d(x^i, x, 1) > 0\}$, (<0), where $i = (i_0,\ldots,i_{k-1})$.

LEMMA 1. *Let* $x^0,\ldots,x^r \in R^k$ *be in general position,* $\beta = (\beta_1,\ldots,\beta_k)$, $|\beta| = r-k$ *and* E_1, E_2 *be neighbouring p-regions with common side contained in* $A := [x^{i_0},\ldots,x^{i_{k-1}}]$, $i = (i_0,\ldots,i_{k-1})$.
Then

$$\Delta_A \big|_{E_2}^{E_1} D^\beta M(x \mid x^0,\ldots,x^r) = \frac{r!}{(k-1)!} \frac{\prod_{l=1}^k d(x^i, e^l, 0)}{\prod_{l \neq i_0,\ldots,i_{k-1}} d(x^i, x^l, 1)},$$

if E_1 *is on the right hand side of* A.

Proof. Assume that x^0 is on the left hand side of A and that $r > k$. Let x be in the interior of E_2 and let it be such that

(i) The half-line $(x^0 + t(x - x^0), t > 1)$ first intersects the side A.

(ii) It intersects different sides $[x^{j_0},\ldots,x^{j_{k-1}}]$ at different points.

Since $r > k$, we have $|\beta| > 0$ whence for some m, $\beta_m > 0$. Set $\beta - e^m = (\beta_1, \ldots, \beta_{m-1}, \beta_m - 1, \beta_{m+1}, \ldots, \beta_k)$. Then Micchelli's formula (11) gives

$$D^{\beta-e^m}M(x \mid x^0, \ldots, x^r) = r\int_1^\infty t^{-2}D^{\beta-e^m}M(x^0 + t(x - x^0) \mid x^1, \ldots, x^r)\, dt$$

$$= r \cdot M(x \mid x^1, \ldots, x^r)$$

$$+ r\sum_l \frac{\begin{pmatrix} D^{\beta-e^m}M(x^0 + t_l^+(x - x^0) \mid x^1, \ldots, x^r) \\ - D^{\beta-e^m}M(x^0 + t_l^-(x - x^0) \mid x^1, \ldots, x^r) \end{pmatrix}}{t_l},$$

where t_l are all the values of t at which the half-line $(x^0 + t(x - x^0), t > 1)$ intersects sides $\{x^{j_0^l}, \ldots, x^{j_{k-1}^l}\}$, $j^l = (j_0^l, \ldots, j_{k-1}^l) \in I_k^r$. If s is sufficiently small then by (ii) the half-lines $(x^0 + t(x - x^0), t > 1)$ and $(x^0 + t(x + se^m - x^0), t > 1)$ intersect the same sides. Therefore

$$\frac{1}{s}\,[D^{\beta-e^m}M(x + se^m \mid x^0, \ldots, x^r) - D^{\beta-e^m}M(x \mid x^0, \ldots, x^r)]$$

$$= r \cdot \sum_l \frac{1}{s}\left(\frac{1}{t_{s_l}} - \frac{1}{t_l}\right)[D^{\beta-e^m}M(x^0 + t_l^+(x - x^0) \mid x^1, \ldots, x^r)$$

$$- D^{\beta-e^m}M(x^0 + t_l^-(x - x^0) \mid x^1, \ldots, x^r)].$$

Since

$$d(x^{j^l}, x^0 + t_l(x - x^0), 1) = d(x^{j^l}, x^0 + t_{s_l}(x + se^m - x^0), 1) = 0,$$

it follows that

$$\frac{1}{s}\left(\frac{1}{t_{s_l}} - \frac{1}{t_l}\right) = -\frac{d(x^{j^l}, e^m, 0)}{d(x^{j^l}, x^0, 1)}.$$

Hence

$$D^\beta M(x \mid x^0, \ldots, x^r)$$

$$= -r\sum_l \frac{d(x^{j^l}, e^m, 0)}{d(x^{j^l}, x^0, 1)}\,[D^{\beta-e^m}M(x^0 + t_l^+(x - x^0) \mid x^1, \ldots, x^r)$$

$$- D^{\beta-e^m}M(x^0 + t_l^-(x - x^0) \mid x^1, \ldots, x^r)].$$

Now by (i) we get

$$\Delta_A \big|_{E_2}^{E_1} D^\beta M(x \mid x^0, \ldots, x^r) = r\,\frac{d(x^i, e^m, 0)}{d(x^i, x^0, 1)}\,\Delta_A \big|_{E_2}^{E_1} D^{\beta-e^m}M(x \mid x^1, \ldots, x^r). \qquad (12)$$

We now pass to the case $r = k$, hence $\beta = (0, \ldots, 0)$.

Since x^0 is on the left hand side of A, $|d(x^i, x^0, 1)| = -d(x^i, x^0, 1)$, it follows that

$$\Delta_A \Big|_{E_2}^{E_1} M(x \mid x^0, x^{i_0}, ..., x^{i_{k-1}}) = \frac{1}{d(x^i, x^0, 1)}. \tag{13}$$

Notice, since $\Delta_A |_{E_2}^{E_1} M = -\Delta_A |_{E_1}^{E_2} M$ the relations (12), (13) hold unchanged if x^0 is on the right hand side of A.

It now remains to combine relations (12), (13). ∎

COROLLARY. *We have the following equality*

$$\Delta_{[x^{i_0},...,x^{i_{k-1}}]} \Big|_L^R M := \Delta_{[x^{i_0},...,x^{i_{k-1}}]} \Big|_{E_2}^{E_1} M = \Delta_{[x^{i_0},...,x^{i_{k-1}}]} \Big|_{E_2'}^{E_1'} M,$$

where E_1, E_2; E_1', E_2' are paris of neighbouring p-regions with common side contained in $[x^{i_0},..., x^{i_{k-1}}]$ and E_1, E_1' are on the right hand side of $[x^{i_0},..., x^{i_{k-1}}]$.

Proof of Theorem 1. Let $\alpha = (\alpha_1,..., \alpha_k)$, $\alpha_m > 0$. On account of the hypothesis of Theorem 1 it is not difficult to get

$$\int_{R^k} D^\alpha f(x)\, M(x \mid x^0,..., x^r)\, dx$$
$$= (-1)^{r-k} \int_{R^k} \frac{\partial}{\partial x_m} f(x)\, D^{\alpha - e^m} M(x \mid x^0,..., x^r)\, dx.$$

Denote by Ω the collection of all p-regions. Then we have

$$\int_{R^k} \frac{\partial}{\partial x_m} f(x)\, D^{\alpha - e^m} M(x \mid x^0,..., x^r)\, dx$$
$$= \sum_{E \in \Omega} D^{\alpha - e^m} M \big|_E \int_E \frac{\partial}{\partial x_m} f(x)\, dx$$
$$= \sum_{E \in \Omega} D^{\alpha - e^m} M \big|_E \cdot \int_E d(f(x)\, dx_1 \wedge \cdots \wedge dx_{m-1} \wedge dx_{m+1} \wedge \cdots \wedge dx_k).$$

The above corollary and Stokes' Theorem give

$$= - \sum_{(i_0,...,i_{k-1}) \in I_k^r} \Delta_{[x^{i_0},...,x^{i_{k-1}}]} \Big|_L^R M(x \mid x^0,..., x^r)$$
$$\cdot \int_{[x^{i_0},...,x^{i_{k-1}}]} f(x)\, dx_1 \wedge \cdots \wedge dx_{m-1} \wedge dx_{m+1} \wedge \cdots \wedge dx_k.$$

To complete the proof, we notice that

$$\int_{[x^{i_0},\ldots,x^{i_{k-1}}]} f(x)\,dx_1 \wedge \cdots \wedge dx_{m-1} \wedge dx_{m+1} \wedge \cdots \wedge dx_k$$
$$= d(x^i, e^m, 0)\, f\{x^i\}.$$

and make use of Lemma 1. ∎

3. Lagrange Interpolation in R^k

THEOREM 2. *Let $x^0,\ldots, x^r \in R^k$ be in general position and γ_i, $i \in I_k^r$, are real numbers.*

Then there exists a unique k-variate polynomial P of total degree not exceeding $r - k + 1$ such that

$$P\{x^i\} = \gamma_i, \qquad i \in I_k^r.$$

Proof. Consider the map

$$\Lambda: \Pi_{r-k+1}(R^k) \to R^N, \qquad N = \binom{r+1}{k},$$

defined by $(\Lambda P)_i = P\{x^i\}$, $i \in I_k^r$.

Since $\dim \Pi_{r-k+1}(R^k) = \dim(R^N) = \binom{r+1}{k}$, it is enough to prove that $(\Lambda P)_i = 0$, $\forall i \in I_k^r$ force $P \equiv 0$.

Indeed from Theorem 1 and $P\{x^i\} = 0$, $i \in I_k^r$, we have $[x^0,\ldots, x^r]^\alpha P = 0$ for all α, $|\alpha| = r - k + 1$.

This means that the total degree of P is $< r - k + 1$. Now we apply Theorem 1 to the points $x^0,\ldots, x^{r-1}$ and similarly get $[x^0,\ldots, x^{r-1}]^\alpha P = 0$ for all α, $|\alpha| = (r-1) - k + 1$ which means that the total degree of P is $< r - k$. Continuing in this way, we finally obtain $P \equiv 0$. ∎

We denote by P_f the above unique polynomial for which

$$P_f\{x^i\} = f\{x^i\}, \qquad \forall i \in I_k^r.$$

This we shall briefly write

$$P_f = f/\{x^0,\ldots, x^r\}.$$

THEOREM 3. *Let $x^0,\ldots, x^r \in R^k$ be in general position,*

$$P_f = f/\{x^0,\ldots, x^r\}, \qquad x \in R^k, \qquad (i_1,\ldots, i_{k-1}) \in I_{k-1}^r.$$

Then

$$f\{x, x^{i_1},..., x^{i_{k-1}}\} = P_f\{x, x^{i_1},..., x^{i_{k-1}}\} + \int_{[x,x^0,...,x^r]} \prod_{\substack{l \neq i_1,...,i_{k-1} \\ 0 \leqslant l \leqslant r}} D_{x-x^l} f \tag{14}$$

or

$$f\{x, x^{i_1},..., x^{i_{k-1}}\} = P_f\{x, x^{i_1},..., x^{i_{k-1}}\} + \sum_{|\alpha| = r-k+2} [x, x^0,..., x^r]^\alpha f(\varphi_\alpha - P_{\varphi_\alpha})\{x, x^{i_1},..., x^{i_{k-1}}\}, \tag{15}$$

where $\varphi_\alpha := x^\alpha := x_1^{\alpha_1} \cdots x_k^{\alpha_k}$.

Proof. Let $0 \leqslant m \leqslant r$, $m \in \{i_1,..., i_{k-1}\}$. Of course

$$\int_{[x,x^0,...,x^r]} \prod_{\substack{l \neq i_1,...,i_{k-1} \\ 0 \leqslant l \leqslant r}} D_{x-x^l} f = \int_{[x,x^0,...,x^r]} \prod_{\substack{l \neq i_1,...,i_{k-1} \\ 0 \leqslant l \leqslant r}} D_{x-x^l}(f - P_f).$$

Then we have by Micchelli's relation (7)

$$\int_{[x,x^0,...,x^m]} \prod_{\substack{l \neq i_1,...,i_{k-1} \\ 0 \leqslant l \leqslant m}} D_{x-x^l}(f - P_f)$$

$$= \int_{[x,x^0,...,x^{m-1}]} \prod_{\substack{l \neq i_1,...,i_{k-1} \\ 0 \leqslant l \leqslant m-1}} D_{x-x^l}(f - P_f)$$

$$- \int_{[x^0,...,x^m]} \prod_{\substack{l \neq i_1,...,i_{k-1} \\ 0 \leqslant l \leqslant m-1}} D_{x-x^l}(f - P_f).$$

Since

$$\int_{[x^0,...,x^m]} \prod_{\substack{l \neq i_1,...,i_{k-1} \\ 0 \leqslant l \leqslant m-1}} D_{x-x^l}(f - P_f) = 0$$

by Theorem 1, the last equality gives (14).

Equation (15) easily follows from (14). It is not difficult to get (15) as well using only the fact that $[x, x^0,..., x^r]^\alpha (f - P_f)$, $|\alpha| = r - k + 2$, is linear combination of

$$(f - P_f)\{x, x^{j_1},..., x^{j_{k-1}}\}, \qquad (j_1,..., j_{k-1}) \in I_{k-1}^r. \quad \blacksquare$$

Formula (14) is similar to the error formula in Kergin's interpolation, obtained by Micchelli and Milman [11, 12]. These formulas are analogous to the following one-dimensional relation

$$f(x) = P(x) + (x - x_0) \cdots (x - x_r)[x, x_0, ..., x_r] f,$$

where P is the unique polynomial of degree $\leqslant r$ such that

$$P(x_i) = f(x_i), \qquad i = 0, ..., r.$$

In the univariate case we have for that polynomial Newton's representation

$$P(x) = f(x_0) + (x - x_0)[x_0, x_1] f + \cdots$$
$$+ (x - x_0) \cdots (x - x_{r-1})[x_0, ..., x_r] f.$$

The next theorem gives us the multivariate analog of this formula. First we shall prove

LEMMA 2. *Let $x, x^{i_0}, ..., x^{i_m} \in R^k$, then*

$$D_{x - x^{i_m}} \int_{[x, x^{i_0}, ..., x^{i_m}]} f = \int_{[x, x, x^{i_0}, ..., x^{i_{m-1}}]} f - \int_{[x, x^{i_0}, ..., x^{i_m}]} f. \tag{16}$$

Proof. Let $0 = y = (1 + s)x - sx^{i_m} - (x + s(x - x^{i_m}))$. Then Micchelli's formula (7) gives

$$0 = -\int_{[x, x + s(x - x^{i_m}), x^{i_0}, ..., x^{i_m}]} D_y f = (1 + s) \int_{[x + s(x - x^{i_m}), x^{i_0}, ..., x^{i_m}]} f$$
$$- s \int_{[x, x + s(x - x^{i_m}), x^{i_0}, ..., x^{i_{m-1}}]} f - \int_{[x, x^{i_0}, ..., x^{i_m}]} f,$$

hence

$$\frac{1}{s} \left[\int_{[x + s(x - x^{i_m}), x^{i_0}, ..., x^{i_m}]} f - \int_{[x, x^{i_0}, ..., x^{i_m}]} f \right]$$
$$= \int_{[x, x + s(x - x^{i_m}), x^{i_0}, ..., x^{i_{m-1}}]} f - \int_{[x + s(x - x^{i_m}), x^{i_0}, ..., x^{i_m}]} f.$$

Now it remains to pass to the limit as $s \to 0$. ∎

From (16) readily follows

$$D_{x-x^{i_m}} \int_{[\underbrace{x,\ldots,x}_{l},x^{i_0},\ldots,x^{i_m}]} f$$

$$= l \cdot \int_{[\underbrace{x,\ldots,x}_{l+1},x^{i_0},\ldots,x^{i_m-1}]} f - l \int_{[\underbrace{x,\ldots,x}_{l},x^{i_0},\ldots,x^{i_m}]} f. \tag{17}$$

Also the analog of (16) for multivariate B-splines is

$$D_{x-x^{i_m}} M(y \mid x, x^{i_0},\ldots, x^{i_m})$$
$$= M(y \mid x, x, x^{i_0},\ldots, x^{i_{m-1}}) - M(y \mid x, x^{i_0},\ldots, x^{i_{m-1}})$$

whenever $\mathrm{vol}_k[x, x^{i_0},\ldots, x^{i_m}] \neq 0$, and where $D_z = \sum_{i=1}^k z_i(\partial/\partial x_i)$.

THEOREM 4. *Let $P_f = f/\{x^0,\ldots, x^r\}$. Then we have*

$$P_f(x) = \sum_{i=k-1}^{r} \sum_{|\alpha|=i-k+1} [x^0,\ldots, x^i]^\alpha f(\varphi_\alpha - P_{\varphi_\alpha})(x), \tag{18}$$

where $\varphi_\alpha = x^\alpha$ and $P_{\varphi_\alpha} = \varphi_\alpha/\{x^0,\ldots, x^{|\alpha|+k-2}\}$, $P_{\varphi_0} \equiv 0$.

Proof. On account of Theorem 3 we have

$$P_{f/\{x^0,\ldots,x^r\}}\{x, x^0,\ldots, x^{k-2}\}$$
$$= P_{f/\{x^0,\ldots,x^{r-1}\}}\{x, x^0,\ldots, x^{k-2}\}$$
$$+ \sum_{|\alpha|=r-k+1} [x, x^0,\ldots, x^{r-1}]^\alpha P_{f/\{x^0,\ldots,x^r\}}(\varphi_\alpha - P_{\varphi_\alpha})\{x, x^0,\ldots, x^{k-2}\},$$

where $P_{f/A}$ is the unique polynomial of the corresponding degree which interpolates f at the set A.

Since

$$[x, x^0,\ldots, x^{r-1}]^\alpha P_{f/\{x^0,\ldots,x^r\}} = [x^r, x^0,\ldots, x^{r-1}]^\alpha P_{f/\{x^0,\ldots,x^r\}} = [x^0,\ldots, x^r]^\alpha f$$

therefore

$$P_{f/\{x^0,\ldots,x^r\}}\{x, x^0,\ldots, x^{k-2}\}$$
$$= P_{f/\{x^0,\ldots,x^{r-1}\}}\{x, x^0,\ldots, x^{k-2}\}$$
$$+ \sum_{|\alpha|=r-k+1} [x^0,\ldots, x^r]^\alpha f(\varphi_\alpha - P_{\varphi_\alpha})\{x, x^0,\ldots, x^{k-2}\}.$$

Similarly we have

$$P_{f/\{x^0,\ldots,x^{r-1}\}}\{x, x^0,\ldots, x^{k-2}\}$$
$$= P_{f/\{x^0,\ldots,x^{r-2}\}} + \sum_{|\alpha|=r-k} [x^0,\ldots, x^{r-1}]^\alpha f(\varphi_\alpha - P_{\varphi_\alpha})\{x, x^0,\ldots, x^{k-2}\}.$$

Continuing in this way, we finally get

$$P_{f/\{x^0,\ldots,x^{k-1}\}}\{x, x^0,\ldots, x^{k-2}\} = [x^0,\ldots, x^{k-1}]f,$$

as can be easily checked.

If we sum up these relations we get

$$P_f\{x, x^0,\ldots, x^{k-2}\}$$
$$= \sum_{i=k-1}^{r} \sum_{|\alpha|=i-k+1} [x^0,\ldots, x^i]^\alpha f(\varphi_\alpha - P_{\varphi_\alpha})\{x, x^0,\ldots, x^{k-2}\}$$
$$= \left[\sum_{i=k-1}^{r} \sum_{|\alpha|=i-k+1} [x^0,\ldots, x^i]^\alpha f(\varphi_\alpha - P_{\varphi_\alpha})\right]\{x, x^0,\ldots, x^{k-2}\},$$

whence by Lemma 2, the proof is completed. ∎

From this theorem we can readily find the polynomial of total degree $\leqslant r-k+1$, $P_{j_0,\ldots,j_{k-1}}$, which for the set $\{x^0,\ldots, x^r\}$ has the property:

$$P_{j_0,\ldots,j_{k-1}}\{x^{i_0},\ldots, x^{i_{k-1}}\} = \delta^{i_0,\ldots,i_{k-1}}_{j_0,\ldots,j_{k-1}} \qquad \text{for all} \quad (i_0,\ldots, i_{k-1}) \in I^r_k.$$

Namely,

$$P_{j_0,\ldots,j_{k-1}}(x) = \sum_{|\alpha|=r-k+1} C^\alpha_{j_0,\ldots,j_{k-1}}(\varphi_\alpha - P_{\varphi_\alpha})(x),$$

where $P_{\varphi_\alpha} = \varphi_\alpha/\{x^0,\ldots, x^r\}\backslash\{x^{j_0}\}$ and $C^\alpha_{j_0,\ldots,j_{k-1}} = C^\alpha_j$ is given as in Theorem 1.

Let $P_{\varphi_\alpha} = \varphi_\alpha/\{x^0,\ldots, x^m\}$, $\varphi_\alpha = x^\alpha$, $|\alpha| = m-k+1$, $(i_1,\ldots, i_{k-1}) \in I^m_{k-1}$. Then, of course, Theorem 3 gives

$$(\varphi_\alpha - P_{\varphi_\alpha})\{x, x^{i_1},\ldots, x^{i_{k-1}}\} = \frac{1}{m!} \prod_{\substack{n \neq i_1,\ldots,i_{k-1} \\ 0 \leqslant n \leqslant m}} D_{x-x^n}\varphi_\alpha. \tag{19}$$

The next lemma gives a striking formula for the value $(\varphi_\alpha - P_{\varphi_\alpha})(x)$.

LEMMA 3. *Let* $P_{\varphi_\alpha} = \varphi_\alpha/\{x^0,\ldots, x^m\}$, $|\alpha| = m-k+1$, $(i_l,\ldots, i_{k-1}) \in I^m_{k-l}$, $2 \leqslant l \leqslant k$. *Then*

$$(\varphi_\alpha - P_{\varphi_\alpha})\{\underbrace{x,\dots,x}_{l}, x^{i_l},\dots,x^{i_{k-1}}\}$$

$$= \sum_{\substack{(j_1,\dots,j_{l-1})\in I_l^m \\ \{j_1,\dots,j_{l-1}\}\cap\{i_l,\dots,i_{k-1}\}=\varnothing}} (\varphi_\alpha - P_{\varphi_\alpha})\{x, x^{j_1},\dots,x^{j_{l-1}}, x^{i_l},\dots,x^{i_{k-1}}\}. \quad (20)$$

In particular for $l=k$,

$$(\varphi_\alpha - P_{\varphi_\alpha})(x) = (k-1)! \sum_{(j_1,\dots,j_{k-1})\in I_{k-1}^m} (\varphi_\alpha - P_{\varphi_\alpha})\{x, x^{j_1},\dots,x^{j_{k-1}}\}. \quad (21)$$

Proof. Let us prove (20) by induction on l. It is not hard to obtain from (19)

$$D_{x-x^{i_1}}(\varphi_\alpha - P_{\varphi_\alpha})\{x, x^{i_1},\dots,x^{i_{k-1}}\} = \sum_{\substack{0\leqslant n\leqslant r \\ n\neq i_1,\dots,i_{k-1}}} (\varphi_\alpha - P_{\varphi_\alpha})\{x, x^n, x^{i_2},\dots,x^{i_{k-1}}\}. \quad (22)$$

Using Lemma 2 we get (20) for $l=2$, namely,

$$(\varphi_\alpha - P_{\varphi_\alpha})\{x, x, x^{i_2},\dots,x^{i_{k-1}}\} = \sum_{\substack{0\leqslant n\leqslant r \\ n\neq i_2,\dots,i_{k-1}}} (\varphi_\alpha - P_{\varphi_\alpha})\{x, x^n, x^{i_2},\dots,x^{i_{k-1}}\}.$$

Assume that (20) holds for $l=l_0$. Hence

$$D_{x-x^{l_0}}(\varphi_\alpha - P_{\varphi_\alpha})\{\underbrace{x,\dots,x}_{l_0}, x^{i_{l_0}},\dots,x^{i_{k-1}}\}$$

$$= \sum_{\substack{(j_1,\dots,j_{l_0-1})\in I_{l_0}^m \\ \{j_1,\dots,j_{l_0-1}\}\cap\{i_{l_0},\dots,i_{k-1}\}=\varnothing}} D_{x-x^{l_0}}(\varphi_\alpha - P_{\varphi_\alpha}) \{x, x^{j_1},\dots,x^{j_{l_0-1}}, x^{i_{l_0}},\dots,x^{i_{k-1}}\}.$$

We now apply (17) and (22) to the left and right hand side, respectively. This gives

$$l_0(\varphi_\alpha - P_{\varphi_\alpha})\{x,\dots,x, x^{i_{l_0+1}},\dots,x^{i_{k-1}}\}$$

$$= l_0 \sum_{(j_1,\dots,j_{l_0-1})\in I_{l_0}^m} (\varphi_\alpha - P_{\varphi_\alpha})\{x, x^{j_1},\dots,x^{j_{l_0-1}}, x^{i_{l_0}},\dots,x^{i_{k-1}}\}$$

$$+ \sum_{\substack{(j_1,\dots,j_{l_0-1})\in I_{l_0}^m \\ \{j_1,\dots,j_{l_0-1}\}\cap\{i_{l_0},\dots,i_{k-1}\}=\varnothing}} \sum_{n\neq j_1,\dots,j_{l_0-1},i_{l_0},\dots,i_{k-1}} (\varphi_\alpha - P_{\varphi_\alpha}) \{x, x^{j_1},\dots,x^{j_{l_0-1}}, x^n, x^{i_{l_0+1}},\dots,x^{i_{k-1}}\}$$

$$= l_0 \sum_{\substack{(j_1,\dots,j_{l_0})\in I_{l_0+1}^m \\ \{j_1,\dots,j_{l_0}\}\cap\{i_{l_0+1},\dots,i_{k-1}\}}} (\varphi_\alpha - P_{\varphi_\alpha})\{x, x^{j_1},\dots,x^{j_{l_0}}, x^{i_{l_0+1}},\dots,x^{i_{l-1}}\}. \quad \blacksquare$$

Let us mention that (21) holds also for φ_α replaced by any polynomial of total degree $\leqslant m-k+1$.

We now obtain the following interesting analog of (18).

COROLLARY. *Let $P_f = f/\{x^0,..., x^r\}$. Then we have*

$$P_f(x) = (k-1)! \sum_{i=k-1}^{r} \sum_{(j_1,\ldots,j_{k-1}) \in I_{k-1}^{i-1}} \times \int_{[x^0,\ldots,x^i]} \prod_{\substack{l \neq j_1,\ldots,j_{k-1} \\ 0 \leqslant l \leqslant i-1}} D_{x-x^l} f. \tag{23}$$

Proof. This readily follows from relations (18), (19) and (21). ∎

In particular if all the points $x^0,..., x^r$ in (23) coincide with x^0 then $P_f(x)$ (as in Kergin interpolation [11, 12]) reduces to the Taylor polynomial of f at x^0:

$$P_f(x) = f(x^0) + D_{x-x^0} f(x^0) + \cdots + \frac{1}{(r-k+1)!} D_{x-x^0}^{r-k+1} f(x^0).$$

The following lemma finds its origin in [10], where a similar result for Kergin interpolation is given. Here we use a weaker hypothesis.

LEMMA 4. *Let $x^0,..., x^r \in R^k$ be in general position,*

$$P_{f_n} = f_n/\{x^0,..., x^r\}, \qquad n = 1,..., m,$$

and

$$\int_{[x^{i_1},\ldots,x^{i_{l+k}}]} \sum_{n=1}^{m} q_n f_n = 0, \qquad \forall (i_1,..., i_{l+k}) \in I_{m+l}^r,$$

where q_n, $n = 1,..., m$, is a constant coefficient homogeneous differential operator of order l. Then

$$\sum_{n=1}^{m} q_n P_{f_n} \equiv 0.$$

Proof. Denote

$$P = \sum_{n=1}^{m} q_n P_{f_n}.$$

First we note that

$$\int_{[x^{i_1},\ldots,x^{i_{l+k}}]} P = \int_{[x^{i_1},\ldots,x^{i_{l+k}}]} \sum_{n=1}^{m} q_n P_{f_n} = 0,$$

since

$$\int_{[x^{i_1},\ldots,x^{i_{l+k}}]} q_n P_{f_n} = \int_{[x^{i_1},\ldots,x^{i_{l+k}}]} q_n f_n$$

by Theorem 1 or by Micchelli's relation (7).

Once more relation (7) implies for all α, $|\alpha| = i-k-l+1$ and i, $l+k-1 \leqslant i \leqslant r$,

$$\int_{[x^0,\ldots,x^i]} D^\alpha P = 0.$$

Indeed, the left hand side is linear combination of

$$\int_{[x^{i_1},\ldots,x^{i_{l+k}}]} P, \qquad (i_1,\ldots,i_{l+k}) \in I^r_{l+k},$$

and since $P \in \Pi_{r-k-l+1}(R^k)$ the proof is complete. ∎

This lemma readily provides the complex analytic version of our interpolation essentially in the same way as in the Kergin case, for which we refer to [9, Sect. 5].

At the end of this part we give some error estimates, which find their origin in [12].

LEMMA 5. *Let $x^0,\ldots,x^r \in R^k$, $P_f = f/\{x^0,\ldots,x^r\}$ and $(i_1,\ldots,i_{k-1}) \in I^r_{k-1}$. Then*

$$|f\{x, x^{i_1},\ldots,x^{i_{k-1}}\} - P_f\{x, x^{i_1},\ldots,x^{i_{k-1}}\}|_{L\infty(K)}$$
$$\leqslant \frac{1}{r!}(d_q(K))^{r-k+1}\left(\sum_{|\alpha|=r-k+1} \frac{(r-k+1)!}{\alpha!}\|D^\alpha f\|^P_{L\infty(K)}\right)^{1/P} \tag{24}$$

and

$$|f(x) - P_f(x)|_{L\infty(K)}$$
$$\leqslant \frac{C_k}{(r-k+1)!}(d_q(K))^{r-k+1}\left(\sum_{|\alpha|=r-k+1} \frac{(r-k+1)!}{\alpha!}\|D^\alpha f\|^P_{L\infty(K)}\right)^{1/P}, \tag{25}$$

where $1/p + 1/q = 1$, $K = [x^0,\ldots,x^r]$ and $d_q(K)$ = diameter of K in l_q, and C_k is constant depending only on k.

Proof. Since the volume of Q^r is $1/r!$, (24) follows from (14). To prove (25) we apply Lemma 2 to (15) k times, and then make use of (19), (20). ∎

Let us mention that (25) essentially is the same as error estimate for Kergin interpolation [12]. Thus the result of Micchelli on interpolating a function which has an analytic extension to a sufficiently large region containing the interpolation points (see Theorem 4 in [12]) holds also in our case.

4. Hermite Interpolation in the Plane

We begin this part with the definition of B-splines on hyperplanes.

Assume that $x^0,..., x^r \in R^m$ lie on some hyperplane L of k dimension, $k \leqslant m$, that is $\mathrm{vol}_k[x^0,..., x^r] \neq 0$, $\mathrm{vol}_{k+1}[x^0,..., x^r] = 0$.

Then we find $y^i \in R^{r+m-k}$, $i = 0,..., r$, such that y^i has first m coordinates as x^i, $i = 0,..., r$ respectively, and

$$\mathrm{vol}_r \sigma \neq 0, \qquad \sigma = [y^0,..., y^r].$$

Now the definition looks as before, i.e., for $x = (x_1,..., x_m) \in L$,

$$M_L(x \mid x^0,..., x^r) = \frac{\mathrm{vol}_{r-k}\{y \in \sigma \mid y_i = x_i,\ i = 1,..., m\}}{\mathrm{vol}_r \sigma}.$$

The relation analogous to (8) in this case is

$$\int_L f(x)\, M_L(x \mid x^0,..., x^r)\, ds = \int_{Q^r} f(v_0 x^0 + \cdots + v_r x^r)\, dv_1 \cdots dv_r,$$

where ds is the volume element in L.

Now we shall restrict ourselves to the plane case.

Let $x^0,..., x^r \in R^2$.

For every x^i, x^j, $x^i \neq x^j$, we define the set $A_{x^i x^j} \subset \{x^0,..., x^r\}$ by: $x^l \in A_{x^i x^j}$ iff one of the following two assertions holds:

(i) $x^l = \lambda x^i + (1 - \lambda) x^j$, $0 < \lambda < 1$.

(ii) $x^l = x^i$ or $x^l = x^j$ and $\min(i, j) \leqslant 1 \leqslant \max(i, j)$.

Let us denote $n(x^i, x^j) = \# A_{x^i x^j}$, where $\#$ denotes the cardinality, and $m(x^k) = \#\{l \mid x^l = x^k\}$ ($m(x^k)$ is the multiplicity of the knot x^k).

Definition. Interpolating parameters for the set $\{x^0,..., x^r\}$ and sufficiently smooth function $f: R^2 \to R$ we define as follows:

$$f_{x^ix^j}=\int_{[A_{x^ix^j}]}\left(\frac{\partial}{\partial n}\right)^{n(x^i,x^j)-2} f, \qquad \text{if} \quad x^i \neq x^j,$$

where $\int_{[A_{x^ix^j}]} \varphi$ is given in (4) and $\partial/\partial n$ is the derivative with normal direction to segment $[x^i, x^j]$. Also

$$f_{x^k}^{\alpha_1,\alpha_2}=D^{(\alpha_1,\alpha_2)}f(x^k), \qquad \alpha_1+\alpha_2 \leqslant m(x^k)-2, \qquad \text{if} \quad m(x^k) \geqslant 2.$$

THEOREM 5. *Let $x^0,\dots,x^r \in R^2$. Then for every collection of numbers*

$$\{\gamma_{ij}, \gamma_k^{\beta_1\beta_2} \mid 0 \leqslant i,j,k \leqslant r, x^i \neq x^j, m(x^k) \geqslant 2, \beta_1+\beta_2 \leqslant m(x^k)-2\}$$

there exists a unique 2-variate polynomial P of total degree not exceeding $r-1$ such that

$$P_{x^ix^j}=\gamma_{ij}, \qquad x^i \neq x^j, \qquad 0 \leqslant i,j \leqslant r,$$

and

$$\frac{\partial^{\beta_1+\beta_2}}{\partial x_1^{\beta_1}\,\partial x_2^{\beta_2}} P(x^k)=\gamma_k^{\beta_1,\beta_2}, \qquad m(x^k) \geqslant 2, \qquad \beta_1+\beta_2 \leqslant m(x^k)-2.$$

Proof. Let us first prove that $[x^0,\dots,x^r]_f^\alpha$, for every α, $|\alpha|=r-1$, is linear combination of interpolating parameters. Micchelli's relation (7) can be used as recurrence relation for multivariate divided difference. This reduces our problem to showing that

$$\int_{[x^{i_1},\dots,x^{i_l}]}\left(\frac{\partial}{\partial n}\right)^{l-2} f$$

is a linear combination of interpolating parameters, where $x^{i_1},\dots,x^{i_l}$ lie on some line L and $\partial/\partial n$ is normal direction to that line. It is not difficult to show that in fact it is a linear combination of the following parameters:

$$f_{x^ix^j}, \qquad x^i, x^j \in L, \qquad \#A_{x^ix^j}=l.$$

Indeed

$$\int_{[x^{i_1},\dots,x^{i_l}]}\left(\frac{\partial}{\partial n}\right)^{l-2} f=\int_L M_L(x \mid x^{i_1},\dots,x^{i_l})\left(\frac{\partial}{\partial n}\right)^{l-2} f\, ds,$$

$$f_{x^ix^j}=\int_{[A_{x^ix^j}]}\left(\frac{\partial}{\partial n}\right)^{l-2} f$$
$$=\int_L M_L(x \mid A_{x^ix^j})\left(\frac{\partial}{\partial n}\right)^{l-2} f\, ds$$

and $M_L(x \mid x^{i_1},\dots,x^{i_l})$ is linear combination of $M_L(x \mid A_{x^ix^j})$, $\#A_{x^ix^j}=l$.

Now the proof is similar to that of the Theorem 2 since the number of interpolating parameters is $\binom{r+1}{2}$. ▮

COROLLARY. *Let $x^0,..., x^r \in R^k$ and let P_f be the polynomial with interpolating parameters corresponding to f.*

Then formulas for P obtained in the Lagrange case hold unchanged.

Proof. This is a consequence of the fact that $[x^{l_0},..., x^{l_i}]^\alpha f$, for all $0 \leqslant l_0,..., l_i \leqslant r$, $|\alpha| = i - 1$ is a linear combination of interpolating parameters of f. ▮

Remark. Hermite interpolation in an arbitrary space R^k and another proof of Theorem 1, which is based only on Micchelli's relation (7), will be presented in [8].

In that paper another natural multivariate interpolation procedure, preserving the pointwise nature of Lagrange and Hermite interpolation, will be given.

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